HOSSAM GHANEM

(31) 4.5 *Summary of Graphical Methods(D)*

Example 13 33 May 6, 2004

[9 points] Let $f(x) = \dfrac{3x^2 - 10x + 3}{(x-1)^2}$

.

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Given that $f^{\backslash}(x) = 4\dfrac{x+1}{(x-1)^3}$ .Find the intervals on which f is increasing or. decreasing and find the local extrema, if any.

(c) Given that $f^{\backslash\backslash}(x) = -8\dfrac{x+2}{(x-1)^4}$. Find the intervals on which the graph of f is concave upward or concave downward and find the points of inflection, if any.

(d) Is the graph of f symmetric with respect to the origin? Justify your answer

(e) Sketch the graph of f

Solution

(a)

$H.A:$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2 - 10x + 3}{(x-1)^2} = 3$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^2 - 10x + 3}{(x-1)^2} = 3$$

$$\therefore \ y = 3 \quad H.A$$

$V.A:$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{3x^2 - 10x + 3}{(x-1)^2} = -\infty$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{3x^2 - 10x + 3}{(x-1)^2} = -\infty$$

$$\therefore x = 1 \quad V.A$$

(b)

$$f^{\backslash}(x) = 4\frac{x+1}{(x-1)^3}$$

$f \ \nearrow$ on $(-\infty,-1) \cup (1,\infty)$

$f \ \searrow$ on $(-1,1)$

		-1		1		
$x-1$	$-$		$-$		$+$	1
$x+1$	$-$		$+$		$+$	-1
	$\oplus$		$\ominus$		$\oplus$	
	$\nearrow$		$\searrow$		$\nearrow$	

$$f^{\backslash}(x) = 0$$

$$x + 1 = 0 \qquad \to \qquad x = -1$$

$$f(-1) = \frac{3(-1)^2 - 10(-1) + 3}{(-1-1)^2} = \frac{3 + 10 + 3}{4} = \frac{16}{4} = 4$$

Maximum local extrema at $(-1, 4)$

(c)

$$f^{\backslash\backslash}(x) = -8 \cdot \frac{x+2}{(x-1)^4}$$

The graph of f CD on$(-\infty, -2)$

The graph of f CU on$(-2,1) \cup (1, \infty)$

		-2		1		
$(x-1)^4$	$+$		$+$		$+$	1
$x+2$	$-$		$+$		$+$	-2
	$\ominus$		$\oplus$		$\oplus$	
	↘		↗		↗	

$$f^{\backslash\backslash}(x) = 0$$

$x + 2 = 0 \quad \rightarrow \quad x = -2$

$$f(-2) = \frac{3(-2)^2 - 10(-2) + 3}{(-2-1)^2} = \frac{12 + 20 + 3}{9} = \frac{35}{9}$$

Inflction point at $\left(-2, \frac{35}{9}\right)$

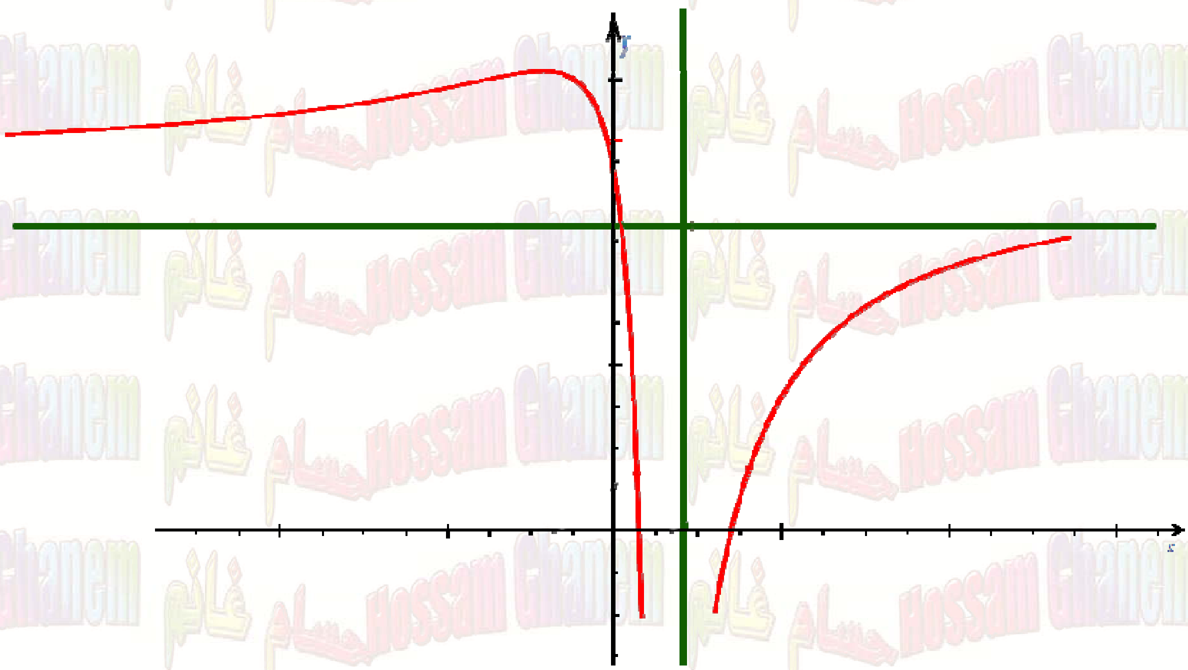
(d)

$$f(x) = \frac{3x^2 - 10x + 3}{(x-1)^2}$$

$$f(-x) = \frac{3x^2 + 10x + 3}{(-x-1)^2} \neq -f(x)$$

The graph of f is not symmetric

(e)



Example 14

34 July 22, 2004

Let $f(x) = x^3 - 6x^2 + 9x - 4$.

(a) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any. (1.5 pt)

(b) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any. (1.5 pts)

(c) Sketch the graph of f. (2 pts)

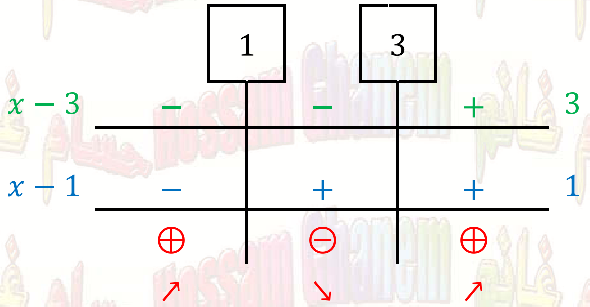
Solution

$f(x) = x^3 - 6x^2 + 9x - 4$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$

$f''(x) = 6x - 12 = 6(x-2)$

(a)



$f'(x) = 3(x-3)(x-1)$

$f \nearrow$ on $(-\infty, 1) \cup (3, \infty)$

$f \searrow$ on $(1,3)$

$f'(x) = 0$

$(x-3)(x-1) = 0 \quad \rightarrow \quad x = 1$ or $x = 3$

$f(1) = 1 - 6 + 9 - 4 = 0$

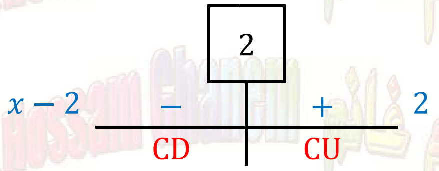
$f(3) = (3)^3 - 6(3)^2 + 9(3) - 4$

$= 27 - 54 + 27 - 4 = -4$

Maximum local extrema at $(1, 0)$

Minimum local extrema at $(3, -4)$

(b)



$f''(x) = 6(x-2)$

The graph of f CD on $(-\infty, -2)$

The graph of f CU on $(2, \infty)$

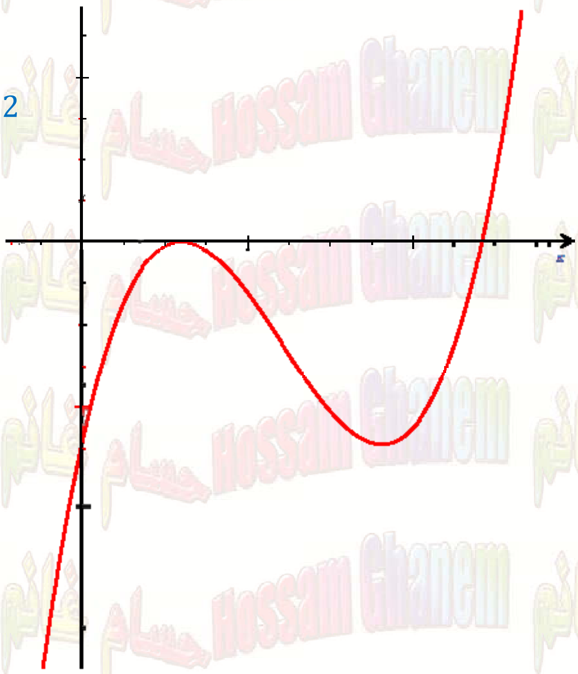
$f''(x) = 0$

$x - 2 = 0 \quad \rightarrow \quad x = 2$

$f(2) = (2)^3 - 6(2)^2 + 9(2) - 4 = 8 - 24 + 18 - 4 = -2$

inflction point at $(2, -2)$

(c)



Example 15

35 December 16, 2004

Let $f(x) = \dfrac{x}{x+1}$

.

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Show that $f^{\backslash}(x) = \dfrac{1}{(x+1)^2}$. Find the intervals on which the graph of f is increasing and the intervals on which the graph of f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f .

(e) Find the maximum and the minimum vales of f on $[\,0\ ,3\,]$ (10 pts.)

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x+1} = 1$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x+1} = 1$$

$\therefore\ y = 1 \quad H.A$

$V.A$:

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \frac{x}{x+1} = \infty$$
$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \frac{x}{x+1} = -\infty$$

$\therefore x = -1 \quad V.A$

(b)

$$f(x) = \frac{x}{x+1}$$
$$f^{\backslash}(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$$

$f^{\backslash}(x) > 0$ on $R\,/\,\{-1\}$

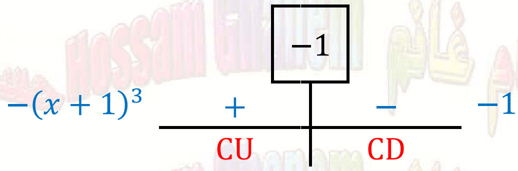
$f \nearrow$ on $R\,/\,\{-1\}$

$f^{\backslash}(x) \neq 0$

No local extrema

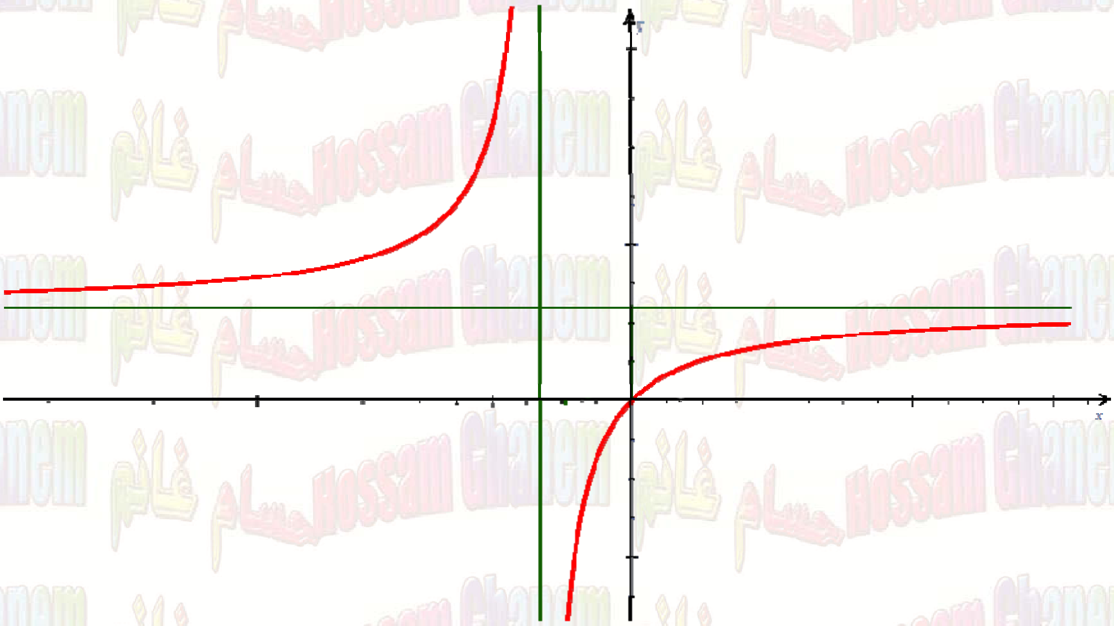
(c)

$$f^{\backslash\backslash}(x) = -2(x+1)^{-3} = \frac{-2}{(x+1)^3}$$



The graph of f CU on $(-\infty, -1)$
The graph of f CD on $(-1, \infty)$

$f^{\backslash\backslash}(x) \neq 0$
No inflction point



(e)

$$f(0) = \frac{0}{0+1} = 0$$

$$f(3) = \frac{3}{3+1} = \frac{3}{4}$$

$\therefore$ absolute Maximum $\frac{3}{4}$ at $\left(3, \frac{3}{4}\right)$
absolute Minimum 0 at $(0, 0)$

Example 16
36 Dec 15, 2005

Let $f(x) = \dfrac{x(3x-8)}{(x-2)^2}$ .and given that $f^{\backslash}(x) = \dfrac{4(4-x)}{(x-2)^3}$ and $f^{\backslash\backslash}(x) = \dfrac{8(x-5)}{(x-2)^4}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(e) Find the maximum and the minimum values of on [3 , 5].

(10 pts.)

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x(3x-8)}{(x-2)^2} = \lim_{x\to\infty} \frac{3x^2-8x}{x^2-4x+4} = 3$$

$$\lim_{x\to-\infty} f(x) = 3$$

$\therefore\ y = 3 \quad H.A$

$V.A$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x(3x-8)}{(x-2)^2} = -\infty$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x(3x-8)}{(x-2)^2} = -\infty$$

$\therefore x = 2 \quad V.A$

(b)

$$f^{\backslash}(x) = \frac{4(4-x)}{(x-2)^3}$$

	2		4		
$(x-2)^3$	$-$		$+$		$+$	2
$4-x$	$+$		$+$		$-$	4
	$\ominus$		$\oplus$		$\ominus$	
	↘		↗		↘	

f ↘ on $(-\infty, 2) \cup (4, \infty)$

f ↗ on $(2, 4)$

$f^{\backslash}(x) = 0$

$4 - x = 0 \qquad x = 4$

$$f(4) = \frac{4(3(4)-8)}{(4-2)^2} = \frac{4(12-8)}{2^2} = \frac{4\cdot 4}{4} = 4$$

Maximum local extrema at (4 , 4)

(c)

$$f^{\backslash\backslash}(x) = \frac{8(x-5)}{(x-2)^4}$$

The graph of f CD on $(-\infty, 2) \cup (2,5)$

The graph of f CU on $(5, \infty)$

	2		5		
$(x-2)^4$	+	+		+	2
$x-5$	−	−		+	5
	⊖	⊖		⊕	
	CD	CD		CU	

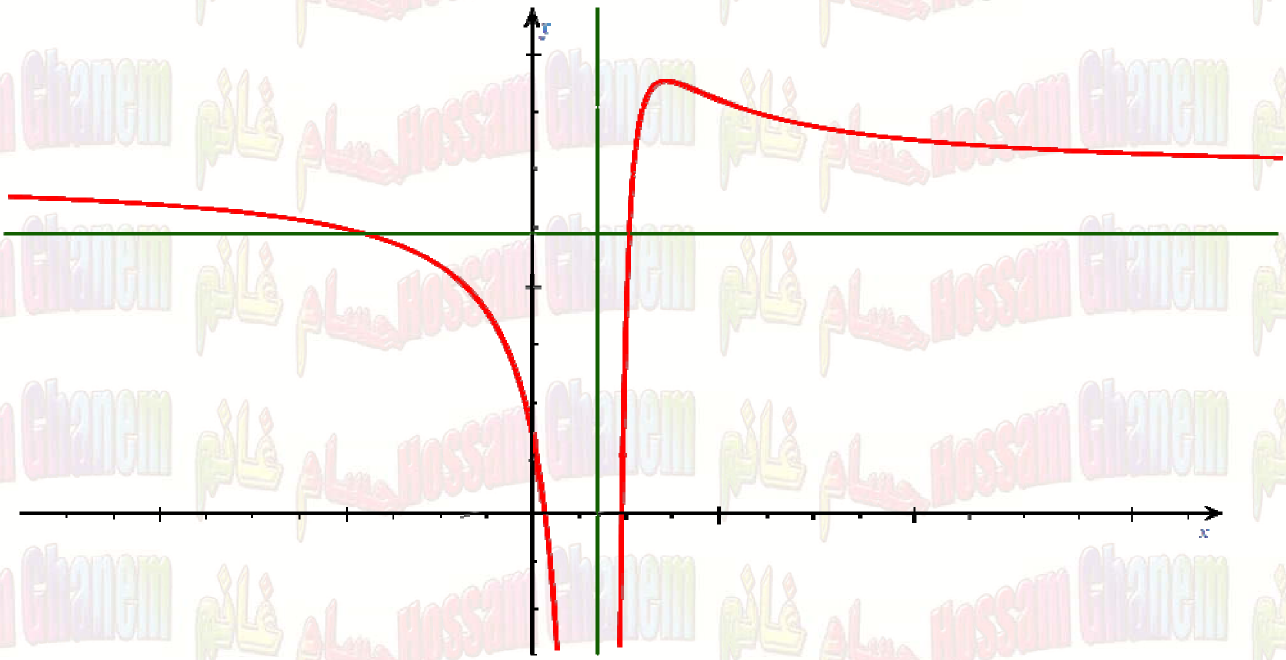
$f^{\backslash\backslash}(x) = 0$

$x - 5 = 0 \quad \rightarrow \quad x = 5$

$$f(5) = \frac{5(3(5)-8)}{(5-2)^2} = \frac{5(15-8)}{3^2} = \frac{5 \cdot 7}{9} = \frac{35}{9}$$

inflction point at $\left(5, \frac{28}{9}\right)$

(d)



(e)

$$f(3) = \frac{3(3(3)-8)}{(3-2)^2} = \frac{3(9-8)}{1} = 3$$

$f(4) = 4$

$$f(5) = \frac{5(15-8)}{9} = \frac{5(7)}{9} = \frac{35}{9} = 3\frac{8}{9}$$

∴ absolute maximum 4 at $(4,4)$

absolute minimum 3 at $(3,3)$

Homework

1 32 December 18, 2003

[4 × 2 points] Let $f(x) = \left(\frac{x}{x+1}\right)^2$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Given that $f^{\backslash}(x) = \frac{2x}{(x+1)^3}$. Find the intervals on which f is increasing or decreasing. Find the local extrema, if any.

(c) Given that $f^{\backslash\backslash}(x) = \frac{2\,(1-2x)}{(x+1)^4}$. Find the intervals on which the graph of f is concave upward or is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

2 27 August 2, 2001

Show that $f(x) = 1 + x - x^2 - x^4$ has no local minimum ?

3 27 August 2, 2001

If $f(x) = 2x^3 - 6x + 11$,
find the maximum and minimum values of f on the interval $[\,0\,,2\,]$

4 40 May 3, 2007

Let $f(x) = 3x\left(x - \frac{5}{3}\right)^{\frac{2}{3}}$ be defined on the interval $[\,-1\,,2\,]$. Find the absolute maximum and absolute minimum of f.

5 Suppose $y = f(x)$ is given by $x^2 + y^2 = 2y$. Find the critical numbers of f

6 28 January 13. 2007

Let

$$f(x) = \begin{cases} x^2 & ,\text{if } \ x \le 1 \\ (2-x)^3 & ,\text{if } \ x > 1. \end{cases}$$

Find the local maxima and the local minima of f.

7 Find the absolute extreme values of $f(x) = x\sqrt{4-x^2}$ for $-1 \le x \le 2$. [4 marks]

8 49 July 24, 2010

(10 Points) Let $f(x) = \dfrac{6x-6}{x^2}$

a. Show that $f'(x) = \dfrac{6(2-x)}{x^3}$ and $f''(x) = \dfrac{12(x-3)}{x^4}$

b. Find the intervals on which f is increasing or decreasing and find the local extrema of f, if any.

c. Find the intervals on which the graph of f is concave up or concave down and find the points of inflection, if any.

d. Find the vertical and horizontal asymptotes of the graph of f, if any.

e. Sketch the graph of f.

9 50 22 December 2010

(4 pts.) Suppose $f(x) = x - 2\sin x$. Find the absolute maximum and minimum values of f on the interval $[0, \pi]$.

10 50 22 December 2010

(8 pts.) Let $f(x) = \dfrac{x-1}{x^3}$ You given that $f'(x) = \dfrac{3-2x}{x^4}$ and $f''(x) = \dfrac{6x-12}{x^5}$

(a) Find the horizontal and verticals asymptotes, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which f is concave upwards and the intervals on which f is concave downwards. Find the inflection points of f, if any.

(d) Sketch the graph of f.

11 51 8 May 2011

[3 pts.] At which points on the curve $y = 10x^3 - 3x^5 + 5$ for $-2 \le x \le 2$ does the tangent line have the greatest slope ?